Math 254B Lecture 19 Notes

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1 Hausdorff Dimension and The Mass Distribution Principle

1.1 Hausdorff measure

Let (X, ρ) be a metric space.

Theorem 1.1. *m_α^* restricts to a measure on $(X, \mathcal{B}_X)$.*

Proof. The reason is that m_α^* is a **metric outer measure**: if $\mathrm{dist}(A, B) > 0$, then $m_\alpha^*(A \cup B) = m_\alpha^* A + m_\alpha^* B$. ☐

Lemma 1.1. *Let $\alpha > 0$. If $\mathcal{H}_\infty^\alpha(A) = 0$, then $m_\alpha^*(A) = 0$.*

Proof. If $\sum_i (\mathrm{diam}(E_i))^\alpha < \varepsilon$, then $\mathrm{diam}(E_i) < \varepsilon^{1/\alpha}$ for all i. So $\mathcal{H}_{\varepsilon^{1/\alpha}}^\alpha(A) < \varepsilon$. As ε is arbitrary, we get $m_\alpha^* A = 0$. ☐

Corollary 1.1. $\dim_H(A) = \inf\{\alpha : m_\alpha^* A = 0\}$.

Lemma 1.2. *If $0 < \alpha < \beta$, and $m_\beta^* A > 0$, then $m_\alpha^* A = \infty$.*

Proof. Assume $m_\alpha^* A < \infty$. For all $\delta > 0$, there is a covering $\bigcup_i E_i \supseteq A$ with $\mathrm{diam}(E_i) \leq \delta$ such thta $\sum_i (\mathrm{diam}(E_i))^\alpha \leq m_\alpha^* A$. Then

$$\mathcal{H}_\delta^\beta(A) \leq \sum_i (\mathrm{diam}(E_i))^\beta \leq \delta^{\beta-\alpha} \sum_i (\mathrm{diam}(E_i))^\alpha = \delta^{\beta-\alpha} m_\alpha^* A,$$

so $m_\beta^* A = 0$. ☐

1.2 Properties of Hausdorff dimension

Proposition 1.1. *Hausdorff dimension has the following properties:*

1. *If $A \subseteq B$, then $\dim_H(A) \leq \dim_H(B)$.*
2. *If $A = \bigcup_{i=1}^\infty A_i$, then $\dim_H(A) = \sup_i \dim(A_i)$.*

3. $\dim_H \leq \underline{\dim_B}$ *(lower box-covering dimension)*

4. *If* $f : (X, \rho_X) \to (Y, \rho_Y)$ *is such that* $\rho_Y(f(x), f(y)) \leq X\rho_X(x, y)^s$ *for* $0 < s \leq 1$ *and* $C > 0$*, then* $m^*_{\alpha/s}(f(A)) \leq C^{\alpha/s} m^*_\alpha(A)$*. In particular,* $\dim_H(f(A)) \leq \dim_H(A)/s$*.*

Proof. For the second statement, $\geq$ holds vt the first ststament. For $\leq$, suppose $\alpha > \sup_i \dim_H(A_i)$. Then $m^*_\alpha A_i = 0$ for all i, so $m^*_\alpha A \leq \sum_i m^*_\alpha A_i = 0$.

For the fourth statement, let $A \subseteq \bigcup_i E_i$ with $\mathrm{diam}(E_i) < \delta$. THen $f(A) \subseteq \bigcup_i fE_i$ with $\mathrm{diam}(fE_i) \leq C\delta^s$ and $\sum_i (\mathrm{diam}(fE_i))^{\alpha/s} \leq C^{\alpha/s} \sum_i (\mathrm{diam}(E_i))^\alpha$. The left hand side is $\geq \mathcal{H}^{\alpha/s}_{C\delta^s} f(A)) \to 0$, and the right hand side is close to $C^{\alpha/s}\mathcal{H}^\alpha_\delta(A)$. $\square$

Remark 1.1. If $f : \mathbb{R}^d \to \mathbb{R}^d$ with $|f(x) - f(x)| = r|x - y|$, then $m^*_\alpha(f(A)) = r^\alpha m^*_\alpha(A)$. Also, in $\mathbb{R}^d$, m^*_α is translation invariant.

1.3 Bounds on fractal dimension: the mass distribution principle

To find an upper bound on any of our 3 notions of fractal dimension, we need to find an efficient cover. To find a lower bound on the dimension, we need to analyze arbitrary covers. But there are tools to do this.

Proposition 1.2 (Mass distribution principle)**.** *Let* (X, ρ) *be metric space, and let* $A \in \mathcal{B}_X$*. If there is a positive finite Borel measure* μ *on* X *such that* $\mu(X \setminus F) = 0$*,* $\mu A > 0$*, and for some* $\delta_0 > 0$*,* $\mu B_\delta(x) \leq x\delta^\alpha$ *for all* $x \in A$ *and* $\delta \leq \delta_0$*, then*

$$\mathcal{H}^\alpha_\delta(A) \geq \frac{1}{c}\mu(A) \qquad \forall \delta < \delta_0.$$

Proof. If $A \subseteq \bigcup_i E_i$ and $\mathrm{diam}(E_i) = \delta_i < \delta$ for some $\delta \leq \delta_0$, assume that $E_i \cap A \neq \varnothing$ for each i. Then $E_i \subseteq \overline{B_{\delta_i}(x_i)}$ for some $x_i \in E_i \cap A$. So $\mu E_i \leq \mu B_{\delta_i}(x_i) \leq c\delta_i^\alpha$. So

$$c \sum_i \delta_i^\alpha \geq \sum_i \mu E_i \geq \mu A. \qquad \square$$

So the problem of lower bounds can be reduced to finding the right measure.

Example 1.1. $\mathcal{H}^d_\delta([0,1]^d) \leq O_d(1)$, so $m_d[0,1]^d < \infty$. To get a lower bound, $\in Leb(B_\delta(x)) \leq C_\delta \delta^d$. $\mathrm{Leb}|_{[0,1]^d}$ satisfies the mass distribution principle at dimension d, so $m_d[0,1]^d < 0$. So $m_d \,\alpha\, \mathrm{Leb}$.

Example 1.2. Let C_α be the middle-α Cantor set. Construct $\mu \in P(C_\alpha)$ as follows:

1. Let $\varphi : \{0,1\}^{\mathbb{N}} \to C_\alpha$, and let $\mu = \varphi_*((1/2, 1/2)^{\times \mathbb{N}})$.

2. Let $\mu[0, x]$ be the corresponding measure with the distribution function equal to the Cantor staircase function.

Either way, $\mu(I) = 2^{-i}$ if I is a basic interval covering C_α at generation i. On the other hand, $|I| = ((1-\alpha)/2))^i$. So for basic intervals,

$$\mu(I) = |I|^{\log(2)/\log(2/(1-\alpha))}.$$

If now J equals any interval that intersects C_α with $0 \leq |J| \leq 1$ pick i such that $(\frac{1-\alpha}{2})^{i+1} \leq |J| < (\frac{1-\alpha}{2})^i$. Then the number of basic intervals of generation i that can intersect J is $\leq O_\alpha(1)$. So

$$\mu(J) \leq O_\alpha(1)|I|^{\log(2)/\log(2/(1-\alpha))} \leq O_\alpha(1)|J|^{\log(2)/\log(2/(1-\alpha))}.$$

So

$$m_{\log(2)/\log(2/(1-\alpha))}C_\alpha > 0.$$

Here is an extension of the mass distribution principle:

Proposition 1.3. *Let A, μ as before. If*

$$\limsup_{n\to\infty} \frac{\mu(B_n(x))}{r^\alpha} \leq c$$

for all $x \in A$, then $m_\alpha A \geq \mu(A)/c$.

Proof. Let $c' > c$, and let $A_n = \{x : \mu(B_r(x)) \leq c' r^\alpha \ \forall r \leq 1/n\}$. Run the previous version. $\square$